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ABSTRACT

This study examined the development of the rational number concept as a ratio. Preliminary to the description of the study is an introduction discussing constructivism and equilibration. The study itself tests whether equilibration theory holds, and if so, what is the nature of its "phases" and whether these are found at each of the "periods" of development. Findings include: development of the ratio concept occurs in stages which can be both chronologically and structurally differentiated and these stages can be seen as resulting from equilibration processes which can be reorganized into two distinct "periods." A detailed description of the instrument, the orange juice test, is included. (MK)

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CONSTRUCTIVISM AS A MODEL FOR COGNITIVE DEVELOPMENT AND (EVENTUALLY LEARNING)

(Orange Juice Experiment)

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CONSTRUCTIVISM AS A MODEL FOR COGNITIVE DEVELOPMENT AND (EVENTUALLY) LEARNING[1]

The development of proportional reasoning in the child and adolescent

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July 1978

INTRODUCTION

The main emphasis of constructivism[3] is on the dual aspect of subject and object intervening in a process of interaction and reciprocal construction.

At equilibrium, the subject grasps the object exhaustively and bears a judgment which is adequate to the whole object.

However, this same object, which is autonomous in the environment, can become more complex. The consequence will be that one part of the object will be grasped by the subject, while the other part is either ignored or interpreted erroneously. This leads to *centration* or *confusion*.

An example is a subject, familiar with natural numbers, who is placed in front of a rational number. He will interpret it in the light of what he "knows". The number 4/9 will be considered "large", while 2/3 is "small". A certain number of "modifications" have to be made to the concept of natural number, in order to fit it (or equilibrate it) to the new "object" which is the rational number.

Equilibration theory is based on the process of adjusting existing schemes to fit more complex objects in the outer world. It is the outer world which unbalances an existing scheme and forces it to evolve. But the process of change and the reconstruction of a new, "magnified" scheme, is the subject's business, and has to do with what is ordinarily called "understanding". When a subject says: *"I do not understand"*, he means that the

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1 This research was supported by a FCAC grant from the Ministry of Education of Quebec.

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3 The term is used in the cognitive-developmental sense.

new object in front of him is too complex for him to adjust, or is presented to him in such a manner as not to enable him to proceed easily to an adjustment. While the expression: *"Now, I understand"* means that the appropriate modifications have been made and the subject is able to integrate the unfamiliar object and play with it adequately.

Piaget (1975)[1] has called "équilibration majorante" the process through which an increased or magnified scheme is constructed, adapted to the new object in the environment.

THE PHASES OF MAGNIFYING EQUILIBRATION

"Magnifying equilibration"comes about in a step-wise process. Let us examine its premises and consider these phases.

1. - Data in the environment are interpreted by the subject, or "assimilated" through identifiable patterns of behavior or "schemes".

2. - Unfamiliar data in the environment cause a "disturbance" in the functioning of the scheme.

3. - The subject reacts to a disturbance in the environment through a process of "compensation".

4. - The mechanism of compensation is not a one-step process, but consists of a succession of identifiable "phases".

5. - Three phases can be described, which are the following:

   i) At the α-phase, the subject "neglects" the disturbance or simply "avoids" it.

   ii) At the β-phase, the subject "modifies" his scheme in order to "assimilate" the new datum.

   iii) At the γ-phase, the subject integrates the new datum in a hierarchical system.

6. - Finally each "period" of development, i.e. sensori-motor, preoperational and operational, is the seat of a complete process of "magnifying equilibration", each period beginning with a phase of nonbalance and terminating with the construction of a hierarchical system.

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[1] PIAGET, J., (1975). *L'équilibration des structures cognitives, problème central du développement.* Paris: Pressen Universitaires de France, translated as: *The Development of Thought.* New York: Viking Press, 1977.

CONSTRUCTION OF PROPORTIONAL REASONING

We shall examine a series of data, obtained on the development of concepts in the child and adolescent, to test:

(1) whether equilibration theory holds, and, if so,
(2) what is the nature of its "phases",
(3) whether we find these at each of the "periods" of development.

The development of proportional reasoning will here be studied. Two distinct findings are made:

a) Development of the ratio concept occurs in stages which can be both chronologically and structurally differentiated.

b) These stages can be seen as resulting from equilibration processes which can be reorganized into two distinct "periods".

The first preoperational period bears on "terms" : a natural number is equilibrated with an inverse generating, in a four-phase process, the concept of 1:1 ratio varying inside its equivalence class.

The second, operational period, bears on "ordered pairs" : the 1:1 ratio is differentiated in an $a:b$ ratio, where terms are independent in magnitude, both in their state and their transformations, generating in a four-phase process, the Common Denominator and Common Factor algorithms.

The "phases"of equilibration, described by Piaget (1975), are seen here to be distinct "stages", structurally defined and imbedded in one another.

Equilibration, inside a "period", takes place in four phases, Piaget's β-phase being subdivised in two, giving the following:

α- phase: centration on known part of a situation, ignoring unknown part.
β_1- phase: assimilation of new part as complement (state-differentiation).
β_2- phase: relation between complementary parts (differentiation of relation or operation).
γ- phase: hierarchical integration.

THE EXPERIMENT

The development of rational number was studied, under its aspect of ratio, in the Orange Juice Test. This was devised in Quebec, and experimented both in individual and group forms. Through a number of Doctor and Master theses, the methodological aspects of developmental research were set down, with methods for differentiating stages, comparing them chronologically, verifying their integrative character, and determining problem-solving strategies at each level (Noelting, Cloutier and Cardinal, 1975)[1]. The results of a group experimentation will be given here. A publication is in preparation.

Instrument: Orange Juice Test.

A test was devised comprising 23 items, where each consisted in comparing the relative orange taste of a mixture, made up of a certain number of glasses of orange juice and a certain number of glasses of water (see Table 1A).

The items were the outcome of a certain number of previous experiments. Items 24 and 25 were later added to make up the final stage IIIB.

Procedure.

Two items are first discussed with the whole group, with explanations given (items I and II, see Figure 1).

Then each child or adolescent is asked to answer the experimental items 1 to 23, first choosing among three possible choices, then explaining why he made his choice.

Correction

Items 1 to 15 are corrected as given by the subject. Items 16 to 25 needed the examination of explanations, in order to eliminate accidental correct answers due to sole centration effects and no operations being put into use.

Sample.

A sample of 321 subjects were tested, from 6 to 16 years of age (see Table 1C). This corresponded to one class per grade level of Elementary

[1] NOELTING, G., CLOUTIER, R. et CARDINAL, G., *Stades et mécanismes dans le développement de la notion de proportion chez l'enfant et l'adolescent.* Rapport de recherche, Université Laval, Québec, 1975.

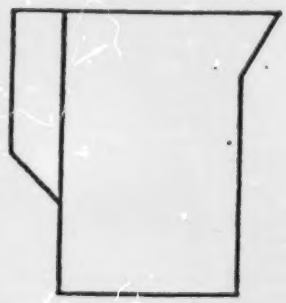
ORANGE JUICE (Form A)

Date _____

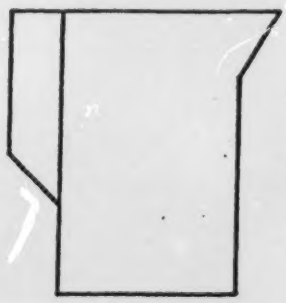
Name _____

Age _____ Date of birth _____

School _____ Class _____



A



B

I

Why ? _____

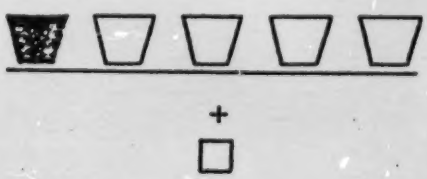
_____

_____

A

II

B



Why ? _____

_____

_____

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FIG. 1. — First page of Orange Juice Test (Group Form A).

TABLE 1A

ITEMS OF ORANGE JUICE TEST (GROUP FORM A)

WITH CORRECT ANSWER AND STAGE

Items	Composition	Correct answer	Stage
I	(3,1) vs. (1,3)	A	IA
II	(1,1) vs. (1,4)	A	IB
1	(1,0) vs. (1,1)	A	IB
2	(4,1) vs. (1,4)	A	IA
3	(1,2) vs. (1,5)	A	IB
4	(1,2) vs. (2,1)	B	IA
5	(1,1) vs. (1,2)	A	IB
6	(3,1) vs. (2,2)	A	IA
7	(1,1) vs. (2,2)	E	IIA
8	(2,3) vs. (1,1)	B	IC
9	(2,2) vs. (3,3)	E	IIA
10	(2,2) vs. (3,4)	A	IC
11	(1,1) vs. (3,3)	E	IIA
12	(1,2) vs. (2,4)	E	IIB
13	(2,1) vs. (3,3)	A	IC
14	(2,3) vs. (1,2)	A	IIIA
15	(4,2) vs. (2,1)	E	IIB
16	(2,1) vs. (4,3)	A	IIIA1
17	(1,3) vs. (2,5)	B	IIIA1
18	(2,1) vs. (3,2)	A	IIIA1
19	(2,3) vs. (3,4)	B	IIIA2
20	(6,3) vs. (5,2)	B	IIIA2
21	(3,2) vs. (4,3)	A	IIIA2
22	(4,2) vs. (5,3)	A	IIIA2
23	(5,2) vs. (7,3)	A	IIIB
24	(3,5) vs. (5,8)	B	IIIB
25	(5,7) vs. (3,5)	A	IIIB

NOTE. — Items 24 and 25, of Stage IIIB, have been added after further experimentation, in order to complete the stage of Higher Formal Operations.

and Secondary Schools. Mathematically advanced classes were chosen at each level, from the same socio-economic level (upper-middle class) of a suburb of Quebec City.

Results.

Items are ordered according to difficulty, then submitted to a Guttman-type scalogram analysis with the help of a computer program (Dixon, 1971)[1]. Satisfactory coefficients were obtained for CR, MMR and PPR, showing that items formed a "perfect" hierarchical scale.

Adjacent items on the scale were then grouped through a process of categorization (Table 1B). Subjects succeeding items of one level, but failing at the next, were considered to make up a "stage". These stages were compared, as to the age distribution of subjects, with the Kolmogorov-Smirnov Test (Siegel, 1956). This is a non-parametric test, as the scale involved is ordinal Adjacent-stage comparison gave significant differences for the last five stages (IC to IIIB). Earlier stages had to be differentiated in an individual experiment (Table IC).

Examination of problems involved at each stage, and strategies used to solve them, led to assign operational levels to these stages, following the Piagetian chronology of development. Typical items of each stage are given in Table 1D.

Explanations subjects give at each stage, for solving the particular problem of the stage, were set in mathematical form (Table 1E). Symbols used are described in the section titled Symbolism.

Finally, the succession of stages was analyzed in terms of equilibration process. Two "periods" of equilibration were found, one corresponding to pre-operational processes leading to the construction of the concept of ratio (Table 1F), the other to construction of the Common Denominator and Common Factor algorithms (Table 1G).

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[1] DIXON, W.G., Ed. *Biomedical Computer Programs*. Berkeley: University of California Press, 1971.

TABLE 1B

ITEMS OF ORANGE JUICE TEST (GROUP FORM A)

ORDERED ACCORDING TO DEGREE OF SUCCESS, THEN CATEGORIZED TO FORM STAGES

Stage	Item	Composition	Frequency of success	Characteristics
0	0	(1,0)vs.(0,1)	-	Differentiation of terms.
IA	2	(4,1)vs.(1,4)	319	Difference between first terms of ordered pairs.
	6	(3,1)vs.(2,2)	319	
	4	(1,2)vs.(2,1)	319	
IB	1	(1,0)vs.(1,1)	311	Like first term, difference between second terms of ordered pairs.
	3	(1,2)vs.(1,5)	307	
	5	(1,1)vs.(1,2)	305	
IC	8	(2,3)vs.(1,1)	295	Equality vs. difference between terms of ordered pairs.
	13	(2,1)vs.(3,3)	291	
	10	(2,2)vs.(3,4)	297	
IIA	9	(2,2)vs.(3,3)	251	(1,1) equivalence class.
	11	(1,1)vs.(3,3)	244	
	7	(1,1)vs.(2,2)	231	
IIB	12	(1,2)vs.(2,4)	186	Any equivalence class.
	15	(4,2)vs.(2,1)	156	
IIIA1	16	(2,1)vs.(4,3)	141	Ordered pairs with two corresponding terms multiple of one another.
	17	(1,3)vs.(2,5)	131	
	14	(2,3)vs.(1,2)	107	
	18	(2,1)vs.(3,2)	88	
IIIA2	20	(6,3)vs.(5,2)	87	Same after simplyfing one pair or extracting (1,1) unit.
	22	(4,2)vs.(5,3)	71	
	19	(2,3)vs.(3,4)	65	
	21	(3,2)vs.(4,3)	59	
IIIB	23	(5,2)vs.(7,3)	51	Any fraction.
	24	(3,5)vs.(5,8)	-	
	25	(5,7)vs.(3,5)	-	

TABLE 1c

COMPARISON OF AGE DISTRIBUTION AT EACH STAGE OF ORANGE JUICE TEST (GROUP FORM A)

Age	N	Stage							
		0	IA	IB	IC	IIA	IIB	IIIA	IIIB
6	14	0	1	2	8	3	0	0	0
7	26	1	1	7	14	2	1	0	0
8	35	1	0	4	12	10	6	2	0
9	43	0	1	2	9	12	13	6	0
10	32	0	0	1	3	13	8	6	1
11	38	0	0	1	5	12	7	9	4
12	34	0	3	1	0	9	5	14	2
13	31	0	2	0	0	2	9	17	1
14	20	0	0	0	1	1	2	10	6
15	29	0	0	0	0	0	8	16	5
16	19	0	0	0	0	1	2	8	8
Total	321	2	8	18	52	65	61	88	27
p[a]		-	-	-	-	<.01	<.01	<.01	<.01
Age of accession[b]		-	-	-	-	8;1	10;5	12;2	(17;0)

NOTES. — [a]Probability level of difference between age distribution of the stage, compared with preceding one, assessed by Kolmogorov-Smirnov Test.

[b]Age of accession to a stage is the age where 50% of Ss solve at least one item of the stage.

TABLE 1D

STAGES IN THE DEVELOPMENT OF THE CONCEPT OF RATIO

(ORANGE JUICE TEST, GROUP FORM A)

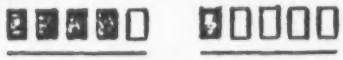
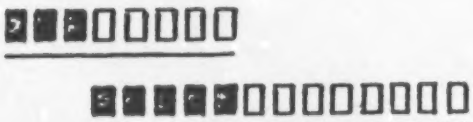
Stage	Name	Age of accession (50% Ss)	Typical item	Characteristics of stage
0	Symbolic	(2;0)	(1,0) vs. (0,1)	Identification of elements.
IA	Lower Intuitive	(3;6)	 (4,1) vs. (1,4)	Comparison of first terms only.
IB	Middle Intuitive	6;4	(1,2) vs. (1,5)	Like first terms, comparison of second terms.
IC	Higher Intuitive	7;0	(3,4) vs. (2,1)	Inverse relation between terms of both ordered pairs.
IIA	Lower Concrete Operational	8;1	(1,1) vs. (2,2)	Equivalence class of (1,1) ratio.
IIB	Higher Concrete Operational	10;5	(2,3) vs. (4,6)	Equivalence class of any ratio.
IIIA	Lower Formal Operational	12;2	(1,3) vs. (2,5)	Ratios with two corresponding terms multiples of one another.
IIIB	Higher Formal Operational	15;10	 (3,5) vs. (5,8)	Any ratio.

TABLE 1E

PROBLEM-SOLVING STRATEGIES AT DIFFERENT STAGES IN THE DEVELOPMENT OF PROPORTIONAL REASONING

Stage	Name	Age of accession (50% Ss)	Typical item (a,b) vs. (c,d)	Strategy
0	Symbolic	(2;0)	(1,0) vs. (0,1)	$a_1 \in A$, $d_1 \in D$
IA	Lower Intuitive	(3;6)	(1,4) vs. (4,1)	$c > a$ Therefore $(c,d) > (a,b)$
IB	Middle Intuitive	6;4	(1,5) vs. (1,2)	$a = c$, $b > d$ Therefore $(c,d) > (a,b)$
IC	Higher Intuitive	7;0	(2,1) vs. (3,4)	$a > b$, $c < d$ Therefore $(a,b) > (c,d)$ even though $a < c$
IIA	Lower Concrete Operation	8;1	(1,1) vs. (2,2)	$m(1,1) = (m,m)$
IIB	Higher Concrete Operation	10;5	(2,3) vs. (4,6)	$m(a,b) = (ma,mb)$ with $a \neq b$
IIIA	Lower Formal Operation	12;2	(1,3) vs. (2,5)	$ma = c$ $m(a,b) = (ma,mb)$ $ma = c$, $mb > d$ $(c,d) > (ma,mb)$ Therefore $(c,d) > (a,b)$
IIIB	Higher Formal Operation	15;10	(3,5) vs. (5,8)	$a+b = g$ $(a,b)g = (a/g,b/g)$ $c+d = h$ $(c,d)h = (c/h,d/h)$ $hg = gh$ $h(a,g) = (ha,hg)$ $g(c,h) = (gc,gh)$ $(gc,gh) > (ha,hg)$ Therefore $(c,d) > (a,b)$

TABLE 1F

THE FOUR STAGES OF EQUILIBRATION OR ADAPTATIVE RECONSTRUCTION

IN THE GENESIS OF EQUIVALENCE CLASSES OF RATIOS

PROBLEM: Working out the relation between terms of a ratio takes place here. The difficulty in understanding the equivalence class of a ratio is differentiating between complementation of terms in the state and covariation of terms in the transformation.
Math. object involved: ordered pair.- Part known: 1st term. - New part: 2nd term

Equilibration stage	Typical item with strategy	Operatory mechanism	Resulting behavior
α-stage Centration by scheme on familiar part of object (1st term).	IA. - (1,4)vs.(4,1) $c < c$ therefore $(a,b)<(c,d)$	Centration on familiar part of object (1st term). New part rejected or treated by same scheme (confusion).	Direct relation between familiar part of object (1st term) and whole (ratio) in all items passed with success.
β_1-stage New part seen as having inverse effect on whole.	IB. - (1,5)vs.(1,2) $a = c,\ b > d$ therefore $(a,b)<(c,d)$	Assimilation of unknown part of object through inversion of scheme.	Oscillation between centration on either 1st or 2nd terms of ratios.
β_2-stage Internal compensation of terms.	IC. - (2,1)vs.(3,4) $a > b,\ c < d$ therefore $(c,d)<(a,b)$ even if $c > a$	Compensation of parts in each object with internal comparison, then conclusion.	Comparison between 1st and 2nd terms in each pair, and conclusion when possible.
γ-stage Hierarchical organization of states in transformation.	IIA.- (1,1)vs.(2,2) Mode C $m(1,1) = (m,m)$ Mode D $1/1 = m/m$	Differentiation between complementation of parts and inversion of scheme. Parts compensate each other (reciprocals) but covary in same direction (complexifying or simplifying ratio).	Complementary parts covary both in direct and inverse directions. Equivalence class of 1:1 ratio.

TABLE 1G

THE FOUR STAGES OF EQUILIBRATION OR ADAPTATIVE RECONSTRUCTION IN THE GENESIS OF THE COMMON DENOMINATOR OR PERCENT ALGORITHM

PROBLEM: At the end of last period, the child has grasped to inverse relationship between terms in the 1:1 ratio.
At this period, he must understand their independence as to size (state) and variation (transformation).

Equilibration stage	Typical item with strategy	Operatory mechanism	Resulting behavior
α-stage Fragmentation between multiplicative and additive parts of (a,b) ratio.	IIA.- i) (1,1)vs.(2,2) $m(1,1)=(2,2)$ (success) ii) (2,3)vs.(4,6) → 2(1,1)+(0,1) vs. 4(1,1)+(0,2)(failure)	Application of (1,1) scheme and centration on excess.	Adequate treatment of 1:1 ratio.
β_1-stage Terms of ratio seen to be independent in state.	IIB.- i) (2,3)vs.(4,6) $m(a,b)=(ma,mb)$ (success) ii) (3,1)vs.(5,2) → (2,1)+(1,0) vs. 2(2,1)+(1,0) (failure)	Differentiation of terms in the state. Application of (a,b) scheme with centration on excess.	Equivalence class of any ratio is grasped. But only conjunctive variation possible.
β_2-stage First coordination between conjunction and disjunction.	IIIA. - (3,1)vs.(5,2) $mb = d$ $m(a,b) = (ma,d)$ $ma > c,\ mb = d$ therefore $(ma,mb) > (c,d)$ whence $(a,b) > (c,d)$	Differentiation of terms in the transformation. Multiplicative conjunction combined with additive disjunction.	Only ratios where corresponding terms are multiple one of another are treated.
γ-stage Hierarchical organization of logical connectives by algebraic operations.	IIIB. - (3,5)vs.(5,8) i) Algebraic addition $a+b = g$ $(a,b) \rightarrow (a,g)$ $c+d = h$ $(c,d) \rightarrow (c,h)$ ii) Algebraic multiplication $hg = gh$ iii) Logical comultiplication $h(a,g) = (ha,hg)$ $g(c,h) = (gc,gh)$ iv) Logical disaddition $ha < gc,\ hg = gh$ $(ha,hg) < (gc,gh)$	Differentiation between logical and algebraic aspects of system. Combinational system along two dichotomies, logical: conjunction vs. disjunction; algebraic: multiplicative vs. additive.	Equivalence class of each ratio integrated by common denominator or common factor. Then additive treatment of numerators. Algorithm of rational number addition is established.

TABLE 1H

COMMON FACTOR ANALYSIS OF ORANGE JUICE EXPERIMENT
(PRINCIPAL FACTORING WITH ITERATION AND VARIMAX ROTATION OF MATRIX).
LOADING OF EACH ITEM IN EACH FACTOR WITH HIERARCHICAL ORDERING OF RESULTS.

(G. Noelting and D. Laveault)

Item		Percentage of succ.		Factors, Eigenvalues and Saturations						Stage
No.	Ident.	%	S-D.	Fact. 5 51.6	Fact. 4 19.2	Fact. 3 9.5	Fact. 2 8.0	Fact. 6 4.2	Fact. 1 3.6	
2	(4,1)vs.(1,4)	98.4	12.4	0.695	0.143	-0.124	0.010	0.022	-0.037	IA
6	(3,1)vs.(2,2)	98.4	12.4	0.555	0.116	0.127	-0.012	0.079	0.044	
4	(1,2)vs.(2,1)	97.2	16.5	0.499	0.007	0.030	0.132	0.056	0.069	
1	(1,0)vs.(1,1)	96.9	17.4	0.302	0.601	0.275	0.132	-0.036	0.048	IB
3	(1,2)vs.(1,5)	93.8	24.2	0.057	0.883	0.207	0.136	0.072	0.123	
5	(1,1)vs.(1,2)	93.5	24.8	0.139	0.617	0.352	0.144	0.115	0.079	
8	(2,3)vs.(1,1)	88.5	32.0	0.049	0.251	0.768	0.166	0.012	0.075	IC
13	(2,1)vs.(3,3)	87.9	32.7	0.040	0.191	0.552	0.315	0.021	0.092	
10	(2,2)vs.(3,4)	87.5	33.1	0.028	0.194	0.672	0.136	0.154	0.108	
9	(2,2)vs.(3,3)	76.0	42.8	0.023	0.060	0.202	0.766	0.044	0.227	IIA
11	(1,1)vs.(3,3)	73.7	45.1	0.085	0.133	0.217	0.793	0.131	0.203	
7	(1,1)vs.(2,2)	70.4	45.7	0.059	0.075	0.154	0.784	0.211	0.257	
12	(1,2)vs.(2,4)	54.8	49.8	0.142	0.075	0.130	0.463	0.531	0.313	IIB
15	(4,2)vs.(2,1)	46.1	50.0	0.105	0.80	0.148	0.355	0.592	0.402	
14	(2,3)vs.(1,2)	31.8	46.6	0.003	0.080	0.073	0.197	0.289	0.639	IIIA
16	(2,1)vs.(4,3)	40.8	49.2	0.081	0.064	0.154	0.262	0.298	0.588	
17	(1,3)vs.(2,5)	36.7	48.2	0.036	0.030	0.040	0.315	0.258	0.556	
18	(2,1)vs.(3,2)	25.2	43.5	0.041	0.029	0.115	0.137	0.166	0.676	
20	(6,3)vs.(5,2)	24.6	43.1	0.019	0.003	0.047	0.165	0.158	0.683	
22	(4,2)vs.(5,3)	21.2	40.9	0.042	0.054	0.045	0.076	-0.003	0.823	
19	(2,3)vs.(3,4)	19.6	39.8	0.030	0.052	0.006	0.120	0.115	0.653	
21	(3,2)vs.(4,3)	17.5	38.0	0.006	0.019	0.064	0.070	-0.078	0.809	
23	(5,2)vs.(7,3)	15.6	36.3	0.025	0.046	0.048	0.082	-0.051	0.674	IIIB

SYMBOLISM

Symbolism of items in algebraic form was introduced to express the strategy common to a stage. A uniform method of placing the two ordered pairs in each item was found necessary, in order to make items comparable. The following rule was applied: the ordered pair with the smaller first term is put first, e.g. (1,4) vs. (4,1). When first terms are equal, the ordered pair with the greater second term is put first, e.g. (1,5) vs. (1,2). With equivalence classes, the pair with the lowest terms thus comes first, allowing multiplicative covariation: 3(1,1) = (3,3). This rearrangement when symbolizing an item will be called the *standardized form*. In the test, the order of pairs in an item is put at random.

The following symbols were adapted to express the sets and subsets of each item. The sets, when placed in the standardized order are called G and H, with g and h expressing their number. The respective subsets of orange juice and water of set G are A and B, with a and b their number. The subsets of set H are C and D, with c and d their number. Thus an item in standardized form remains (a,b) vs. (c,d). Individual elements of a subset such as A are called a_1, a_2, etc. Various operators are introduced by subjects modifying each term of the ordered pairs, i.e. a, b, c and d. The symbols f, j, m and n will be used to denote natural numbers (excluding zero).

QUALITATIVE DESCRIPTION OF STAGES - In choices: A means first pair, B second, E equality. Space does not allow to give examples of each stages.

Examples of behavior of some characteristic stages are given.

Stage IA: Lower intuitive. Centration on the first terms of the ordered pairs.

Success at items such as (1,4)vs.(4,1) and (1,2)vs.(2,1).- The child compares the number of glasses of orange juice in both pairs, or opposes predominance of juice in one pair and water in the other.

Examples of success:

Diane, 4;0	Item B4: (1,2)vs.(2,1) Chooses B (success).	*"Here there is more orange".*
Nathalie, 5;0	Item B3: (4,1)vs.(1,4) Chooses A (success).	*"Because there is a lot of orange juice and only one glass of water".*
Gilles, 4;0	Item B4: (1,2)vs.(2,1)	*"There is a lot of orange juice [illegible]*

Examples of failure:

i) Globalism

France, 5;0	Item B6: (1,0)vs.(1,1) Chooses B (failure).	*"Because there are many".*

ii) Centration

Louis, 4;7	Item D6: (1,1)vs.(1,0) Chooses E (failure).	*"It will taste the same because there is one glass of orangeade there (B) and one glass of orangeade there (A)".*

Stage IIA: Lower concrete operation. Equivalence class of ratio (1,1).

Success at items such as: (1,1)vs.(2,2) and (2,2)vs.(3,3).

Examples of success:

i) Covariation (Mode C)

Johanne, 11;0	Item A7: (1,1)vs.(2,2) Chooses E (success).	*"Each glass dilutes one glass So A has one glass of juice a B has two; A has one glass of water and B two. They are eq only there is more liquid mix in B".*

ii) Division (Mode D)

Martine, 8;0	Item B12: (2,2)vs.(3,3) Chooses E (success).	*"Two for two, here (A); three for three, here (B)".*

Subjects differentiate between state and transformation. The relation between complementary terms in the pair is stabilized as an "invariant". The relation between corresponding terms between pairs is mobilized as a transformation (either co-multiplication or co-division). This yields the simplest equivalenc class, the 1:1 ratio. Strategy for the divisive mode is $m/m = n/n$, correspondi to transposition of a ratio. Strategy for the multiplicative mode is $m(1,1) = (m,m)$; $(m,m)/m = (1,1)$, corresponding to complexifying or simplifying a ratio.

However ratios, where terms are not equal, are failed.

Two modes of behavior are distinguished and will be found at each stage:

Mode C or covariation - Mode D or internal division.

Examples of failure:

i) Centration on the residual after (1,1) covariation (thus using strategy of the stage)

Louise, 11;0	Item A12: (1,2)vs.(2,4) Chooses A (failure).	*"Because the left side has on glass of water more, while th right side has two of them more".*

ii) Centration on either juice or water (repression to earlier strategy)

Diane, 8;0	Item A12: (1,2)vs.(2,4) Chooses A (failure).	*"It is that there are less glasses of water".*

Stage IIIB: Higher formal operation. Common Denominator and Percentage Algorithms.

Examples of success:

i) Common Denominator (Mode C)

Sylvie, 14;0	Item A19: (2,3)vs.(3,4) Chooses B (success).	*"At the right, there is 3/7 of juice for 4/7 of water, that is 15/35 of juice; at t left there is only 14/35".*

ii) Percentage (Mode D)

Réjean, 13;0	Item A23: (5,2)vs.(7,3) Chooses A (success).	*"A = 71 3/7 % because 5/7 ora juice.* *B = 70% because 7/10 orange juice".*

Characteristics of stage IIIB: Differentiation between logical and algebraic transformations, with hierarchical integration.

At stage IIIB, a combinatorial system is formed, where algebraic and logical transformations are differenciated and integrated. These are defined as follows:

Algebraic:

A *binary operation on elements* is an operation in set which combines two elements of the set into a third element of the set.

The addition of two terms a and b of a ratio to find their sum g is a binary addition. The terms of a ratio are considered here as natural numbers with their sum a natural number.

The multiplication of two denominators to find their product is a binary operation. The operation of join or meet on two denominators to find their LCM or HCF is also a binary operation.

Logical:

A *binary operation on propositions* is a connective introduced on two propositions. We shall consider a binary operation on elements as a proposition. Thus the coordinated multiplication or division of both terms of an ordered pair, to find an equivalent, will be considered a conjunctive operation. The isolated multiplication of one term of a ratio or fraction will be considered a disjunctive operation.

Binary operation on terms : $a+b = g$
$g \cdot h = gh$

Binary operation on propositions:

Two propositions: $a \to ma$, $b \to mb$

Conjunction: $(a,b) \to (ma,mb)$ equivalence class

Disjunction: $(a,b) \to (ma,b))$ operation on rational

This distinction is best summarized in the following table :

	Algebraic (terms)	Logical (pairs)
Additive	I $a + b = g$	III $(ha,hg) \gtrless (gc,gh)$
Multiplicative	II $h \times g = hg$	IV $(a,b)/g = (a/g,b/g)$ $h(a,g) = (ha.hg)$

PEDAGOGICAL CONCLUSIONS

I. - *Specific to the proportion concept.*

(1) The concepts of ratio between quantities (e.g. 2 glasses of orange juice for 3 glasses of water), fraction of a set (e.g. 2 glasses of orange juice for 5 glasses of liquid) and fraction of a unit (e.g. 2/5 juice in each glass) should be carefully distinguished.

(2) Proper (< 1) and improper (< 1) fractions should be worked upon simultaneously, e.g. 1/2, 2/4 ..., 2/1, 4/2 ...
Improper fractions should be considered as rational numbers without immediate retrieval of the unit.

(3) From level IIA onwards, any fraction should be envisaged under both its internal division aspect (mode D) and covariation aspect (mode C).

(4) The passage from equivalence of unit fractions, to equivalence of any fraction, should be considered a difficult step, and taking up many years. The child must here differentiate between independence of terms in the state and covariation in transformation. This is the proper problem of the elementary school. Equivalences of 3/5 and 5/7, for instance, in concrete situations, are still considered difficult at the end of the elementary school.

(5) A lot of time should be devoted to problems like 1/3 + 2/9. The multiplicative relation between denominators should be discovered by the pre-adolescents themselves. We find here a combination of the covariation of terms in the equivalence class and "disvariation" of terms in the addition of fractions with like denominators. This is a coordination of conjunction and disjunction applied to the same content. It is characteristic of the formal level of thinking. It is abstract thinking, an operation (additive operation) on an operation (equivalence).

(6) Common denominator (mode C), and reduction to unit (mode D), should be seen as inverse strategies used to liken different denominators. Percentage should be seen as a way of better expressing the ratio to a unit.

II. *Relative to the concepts of periods and phases of equilibration.*

(1) Respect of periods of equilibration i.e. concrete and formal modes of thinking.

A sharp distinction must be made between concrete operations and formal operations. It is a distinction between operation on terms and operation on operations. In period I, operation is performed on data themselves (i.e. *a, b, c, d*). In period II, operation is performed on data constructed from the data (e.g. *ma, a/b,* etc.). This differentiation is especially important to make for teachers in Grade 6, when children are at the frontier of concrete and formal thinking. Some problems, though intricate, are easy because they involve only concrete data. Others, though apparently much simpler, are difficult for the pre-adolescent, because they involve simplification, equivalence, which seems automatic to an adult, but involve retention of constructed data, then operation on these.

(2) Equilibration should be made of what the child knows, to the unknown which is brought to him. In particular the new variable introduced at each period should be clearly identified by the teacher and related to existing schemes in the child.

III. *Conclusions of a general nature bearing on mathematics.*

(1) Emphasis should be put on laying strong foundations rather than rapid but evanescent techniques. Notions should be constructed in their hierarchical order. Motivation is kept up by the process of discovery and construction, and by varying the content for a same structure.

(2) Equilibration - The novel aspect introduced by equilibration theory is the constant interplay between interaction and construction. The dialectical process of uncovering new data is constantly related to the process of structuring the data inside a coherent whole where the new and the old are interrelated. Equilibration to novelty is

related to reorganization of internal structure. Growth consists in being open to the world, but proceeding with system. This opens up a new field of study: the psychology of mathematical construction.

(3) At all levels concrete problems should be worked upon in parallel to symbolic representation of problems.

(4) Stage IIIB, in proportional reasoning, is characterized by the combination of logical reasoning and algebraic operations. The difference between the nil transformation, characteristic of logic (e.g. $p\bar{q}$) and the inverse transformation, characteristic of operations (e.g. 2 + 3 = 5; 5 - 3 = 2) should be made much earlier. Usually a nil transformation is introduced in a combinatorial setting at the formal level: e.g. $pq \vee p\bar{q} \vee \bar{p}q \vee \bar{p}\bar{q}$. This should be prepared at the concrete level (elementary school) by introducing the difference between constancy and variation. Their combination in the constancy-variation scheme is already put into use at stage IB (middle intuitive: 5-6 years of age). The very important role of the "agreement and difference" principle (basis of scientific reasoning as set forth by Roger Bacon and John Stuart Mill) render the early introduction of the constancy-variation scheme imperative. The "all other things being equal" principle is the basis of organized thought at all levels.

(5) Axiomatics vs. constructivism. - Disinterest for mathematics on the part of the layman and the child is due to the obsessive character of axiomatics in contrast with the creative quality of constructivism. Mathematics should become constructive, with emphasis on process, instead of obsessive, with emphasis on structure. Time is ready for a change in attitude in mathematics.

(6) "Intuition" plays a certain part in mathematics, but is never formalized. Cognitive-developmental theory rejets "intuition" and replaces it by "actions" of subjects upon the environment. These actions are reversible and are formalized as "operations". Rules should arise from the nature of mathematical objects upon which activity is performed and their constraints and not from the "axioms" which are elaborated as end products.